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THIN-WALLED ELASTIC SHELLS ANALYSED BY A RAYLEIGH METHOD

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Abstract—Rayleigh successfully analysed inextensional deformation of thin elastic shells by using a simple energy method. Subsequent workers seem to have been put off from using similar methods for shells which suffer extensional as well as bending deformations by the fact that the calculations get messy. In this paper we develop the kinematic relation between surface strains and changes in Gaussian curvature, and show that this is a very convenient tool for use in energy calculations. We give two examples of energy calculations for shells loaded by point forces. We find that once the energy expressions have been set up, certain analogies with simpler and already-solved problems become obvious. This leads to simple solutions. A feature of the method is that physically important quantities are not obscured, and distinct regimes of structural action are clearly delineated.

NOTATION

- a radius of cylindrical and spherical shells
- B bending stiffness of analagous beam (force.length^2) (Fig. 5)
- b half-wavelength in longitudinal direction
- c half-wavelength in circumferential direction
- D $= Et^3/12(1-\nu^2)$, bending stiffness
- E Young's modulus
- L length characterising mode of deformation
- l half-length of cylindrical shell (Fig. 4) half-length of beam on elastic foundation (Fig. 5)
- m foundation stiffness (force length^{-2}) (Fig. 5)
- n mode number in Fourier series
- P force applied to shell (Fig. 4)
- Q point force applied to beam (Fig. 5)
- q_n amplitude of line load (force length^{-1}) (Fig. 4)
- R radius of surface
- R_1, R_2 principal radii of surface
- t thickness of shell
- U strain energy
- u displacement component in axial direction
- v displacement component in circumferential direction
- w displacement component in normal direction
- x cartesian co-ordinate; axial co-ordinate
- y cartesian co-ordinate

Greek symbols

- α angular defect, Fig. 1(a)
- α_1, α_2 Lamé parameters, eqn (1)
- γ shear strain
- δ radial displacement of point of application of load on shell
- ϵ direct strain
- η dimensionless co-ordinate, $= y/L$
- θ circumferential co-ordinate
- K Gaussian curvature
- κ_i curvature changes (variously $(1,2)$, (x,y), (x,θ) co-ordinate systems)
- λ $= lt^{1/2}/a^{3/2}$
- ν Poisson's ratio
- ξ dimensionless co-ordinate, $= x/L$
- ξ_1, ξ_2 curvilinear co-ordinates, eqn (1)
- ψ dimensionless group, $= l\sqrt{2}(m/B)^{1/4}$
- $'$ denotes differentiation with respect to x

RAYLEIGH'S METHOD AND SHELL STRUCTURES

In one of the first papers written on the subject of shell structures Rayleigh[1] analysed the natural frequencies of vibration of a thin hemispherical bowl by considering the energy which the bowl would have if it deformed inextensionally. It is therefore surprising that in the subsequent development of shell theory relatively little use has been made of "Rayleigh's method" for analysing the dynamical, or indeed the statical, behaviour of thin elastic shells. The method has been used for analysis of shell structures by only a few workers (e.g. Den Hartog[2]) and it commands only a few pages in the texts on shells by Timoshenko[3], Flügge[4] and Novozhilov[5].

The main reason for the non-proliferation of Rayleigh's early work seems to be that although the method works economically for *inextensional* deformations of shells, it seems to be much more cumbersome when extension of the surface, as well as bending, is taken into account. Since relatively few practical shell structures admit of inextensional modes, Rayleigh's original method does not seem to be useful.

The object of the present paper is to re-habilitate Rayleigh's method for thin elastic shells. The main problem is to set up suitable modes of displacement involving extension of the surface. We tackle this by using a "well-known" purely kinematic relation between certain derivatives of the surface extensional strain components and the change in Gaussian curvature. This makes it possible to work in terms of displacement modes defined only by the component of the displacement normal to the surface, in some practically important examples at least.

In both of the examples which we use to illustrate the method we find that the resulting strain energy expressions are exactly analogous to energy expressions for somewhat simpler problems, viz beams and plates on elastic foundations. As the solutions of these problems are given in well-known texts (e.g. Timoshenko and Woinowsky-Krieger[3], Hétenyi[6]) we obtain some very simple solutions. In other words our use of Rayleigh's method enables us to strike useful analogies between shell problems on the one hand and simpler and already-solved beam and plate problems on the other.

Rayleigh's method is of course always open to the objection that it is approximate, and relies heavily on the intelligent choice of "trial modes", in contrast to the systematic and rigorous methods which are the descendents, so to say, of Love's[7] early work on shells. We hope to show that our up-dated Rayleigh method produces some valuable results for the engineer. In particular the method picks out distinct regimes of structural behaviour and gives the boundaries between the regimes in terms of appropriate dimensionless groups (see Fig. 7). This is of course precisely the kind of information which structural engineers need.

Throughout this paper we shall use the term "Rayleigh's method" to describe that well-known family of approximate "energy" methods for elastic structures which is based on postulated displacement modes. All of our examples involve statical problems, but in a subsequent paper we shall give examples of dynamical behaviour of shells. We shall assume conditions of small displacements and small strains throughout.

THE IMPORTANCE OF GAUSSIAN CURVATURE

It is well-known that during purely inextensional deformation of a surface by bending the Gaussian curvature at a given point on the surface remains constant. We expect therefore that in a deformation of the surface which involves non-zero extensional strains in the surface, the Gaussian curvature will change. Indeed, there should be a direct relation between change of surface strain and change of Gaussian curvature. This kind of relation can be found (sometimes thinly disguised) in texts on differential geometry (e.g. Weatherburn[8]) and in books and papers on shells. We give below the relation for a general surface, as derived by Sanders[9], with the L.H.S. re-arranged:

$$\alpha_1\alpha_2\delta K = \frac{\partial}{\partial\xi_1}\left[\frac{1}{\alpha_1}\left(-\frac{\partial\alpha_2\epsilon_{22}}{\partial\xi_1}+\frac{\partial\alpha_1\epsilon_{12}}{\partial\xi_2}+\frac{\partial\alpha_2}{\partial\xi_1}\epsilon_{11}+\frac{\partial\alpha_1}{\partial\xi_2}\epsilon_{12}\right)\right]$$
$$+\frac{\partial}{\partial\xi_2}\left[\frac{1}{\alpha_2}\left(-\frac{\partial\alpha_1\epsilon_{11}}{\partial\xi_2}+\frac{\partial\alpha_2\epsilon_{12}}{\partial\xi_1}+\frac{\partial\alpha_1}{\partial\xi_2}\epsilon_{22}+\frac{\partial\alpha_2}{\partial\xi_1}\epsilon_{12}\right)\right]. \quad (1)$$

Here ξ_1, ξ_2 are curvilinear co-ordinates, α_1, α_2 are the Lamé parameters appearing in the first fundamental form, and ϵ_{ij} are the surface extensional strains. δK is the change of Gaussian curvature on account of the strains.

In the present paper we shall be concerned mainly with cylindrical shells which, in the undistorted configuration, have zero Gaussian curvature. For such shells eqn (1) simplifies considerably, particularly if we have a rectangular cartesian co-ordinate system within the surface.

It is instructive to derive this special form of eqn (1) from first principles, using the approach to Gaussian curvature described by Hilbert and Cohn-Vossen[10]. Just as the "ordinary" curvature of a plane smooth curve may be defined at any point as the rate of change of angle with arclength (where by "angle" we mean inclination of the tangent to a datum direction), so the Gaussian curvature of a smooth three-dimensional surface may be defined at any point as the rate of change of solid angle with surface area. We find the solid angle subtended by a region of the surface by mapping the edge of the region onto a unit sphere so that corresponding points on the two surfaces have parallel normals, and finding the area enclosed on the sphere.

There are some remarkably simple results in connection with the solid angle subtended by certain surfaces. For example, suppose we construct a cone by cutting out from a flat sheet a sector of angle α (radians) and joining the edges together, as shown in Figs. 1(a) and (b). It is a trivial exercise to show that the solid angle subtended by any closed path encircling the vertex is precisely α: every path of this kind on the cone maps to a circle enclosing area α on the surface of the unit sphere. Now suppose that the same conical surface is turned into a polygonal vertex by flattening the sheet, putting in radial creases (Fig. 1(c)) and re-connecting as in Fig. 1(d) by folding along the creases. Each plane facet now maps to a point on the unit sphere, and each crease to an arc of a great circle, but in fact the area enclosed on the unit sphere remains at exactly α. The proof of this is sketched in Hilbert and Cohn-Vossen[10].

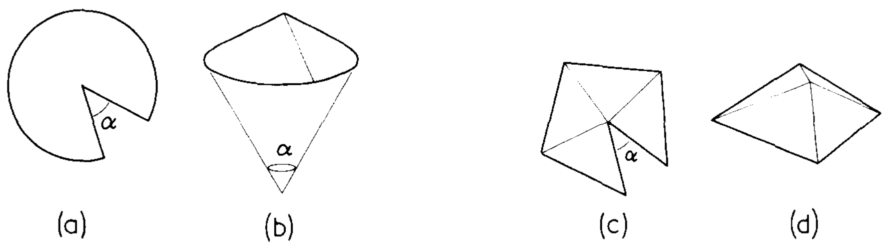

Fig. 1. A "vertex" made by folding a flat sheet subtends a solid angle equal to the "angular defect" of the sheet. The solid angle is not affected by the insertion of creases.

In other words the solid angle subtended by the "cone" is unchanged by inextensional deformation of the cone. This is closely related to the invariance of Gaussian curvature to inextensional deformation: indeed the Gaussian curvature of the cone consists entirely of a singularity at the vertex.

Consider next an arbitrary simply closed polyhedron. At each vertex we define the "angular defect" as 2π—(sum of interior angles of faces meeting at the vertex). Thus, if the plane figure of Fig. 1(c) were folded into a vertex the angular defect would be just α. It follows that since the polyhedron encloses a solid angle of 4π, the sum of the angular defects for all vertices is exactly 4π. This result may also be obtained easily by using Euler's theorem relating the numbers of faces, edges and vertices or an arbitrary simple polyhedron ($f + v = e + 2$). This result is directly useful in the design of near-spherical triangulated polyhedra as used for Radar-enclosing domes, etc: all that is necessary is to distribute the angular defect uniformly over all vertices, if the triangular faces have nearly the same area, or more exactly, to make the angular defect at each vertex proportional to the mean area of the faces meeting at the vertex. Since the *total* angular defect for a complete sphere is always 4π, the angular defect at each vertex decreases as the number of faces increases. As the number of faces becomes very large, the polyhedron approximates more closely to a spherical surface.

Consider next the problem of applying nearly-square flat facets to a doubly-curved surface. This problem must be solved, e.g. in making a plywood formwork for a doubly-curved concrete shell roof. It will be sufficiently general to consider the application of plane, but slightly distorted, squares of side c to a surface of Gaussian curvature $1/R^2$ (where $R \gg c$), for example a portion of a sphere of radius R.

Figure 2(a) shows a patch of 25 squares with which we start. To achieve the required approximation we must simply arrange an angular defect equal to c^2/R^2 at each vertex. A simple scheme for doing this is shown in Fig. 2(b): as we work out from the centre we must shear the

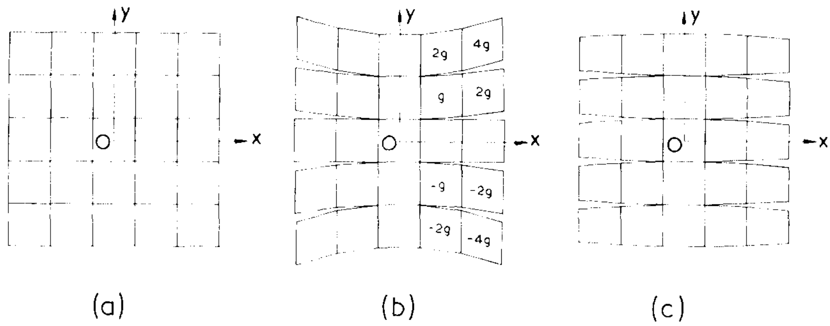

Fig. 2. An approximation to Gaussian curvature may be imparted to a flat sheet (a) by distorting "squares", as in (b) and (c) so that there is a suitable angular defect as each vertex. In scheme (b) the squares are sheared, and the (small) angles of shear are as marked. If the original squares are $c \times c$, and K is the required Gaussian curvature, $g = Kc^2$.

squares in such a way that the angular defect is equal at each vertex. The angle of shear found in this way is indicated on each "square".

A second simple scheme for achieving the same result is shown in Fig. 2(c): here the squares have been distorted into trapezia. In this case of course the areas of the faces furthest from the centre are appreciably smaller than c^2 and we should, strictly, allow for this in deciding on the angular defect. But for only a few squares and $c \ll R$, the effect is negligible.

On Fig. 2 we have marked x and y axes, and it is easy to verify that the pattern of shearing in Fig. 2(b) conforms to

$$\gamma_{xy} = xy/R^2$$

while the pattern of y-direction stretching in Fig. 2(c) conforms to

$$\epsilon_y = -x^2/2R^2.$$

Here the symbols γ and ϵ for shear and direct strain respectively, have the usual small-strain definitions, and x, y are co-ordinates of the midpoints of the "squares". Note that the unit size c does not enter these expressions. In the limit $c \to 0$ we obtain the following expressions for continuous distortion of a flat sheet in order to apply it to a doubly-curved surface:
by shearing:

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{1}{R^2}$$

or by y-direction stretching;

$$\frac{\partial^2 \epsilon_y}{\partial x^2} = -\frac{1}{R^2}$$

or, similarly, by x-direction stretching;

$$\frac{\partial^2 \epsilon_x}{\partial y^2} = -\frac{1}{R^2}.$$

Clearly we can also proceed by a linear combination of these three prescriptions: so we obtain, finally

$$-\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} - \frac{\partial^2 \epsilon_y}{\partial x^2} = K \tag{2}$$

where K is the local Gaussian curvature of the surface. If the surface remains plane, then $K = 0$ and (2) becomes the well-known equation for compatibility of strain in a plane.

Equation (2) is certainly correct for the distortion of a small part of the x–y plane into a curved surface. Unfortunately this method of derivation is inadequate for dealing with the more general problem of distortion of an originally *curved* surface: here an x–y co-ordinate system would be unsuitable, and we need the general version of (2), already given in (1). It is easy to check that (1) reduces to (2) for a cartesian co-ordinate system.

In the case of a *cylindrical* surface, which has zero Gaussian curvature initially, eqn (2) holds intact if the x, y co-ordinate system is used in the obvious way with (say) the x-axis in the direction of the generators. It can be shown that we may also use (2) in the case of a *shallow* shell which touches the x–y plane at the origin.

The main practical point of the present paper is that eqn (1) (or (2) if appropriate) is precisely the kinematic, or compatibility, equation one needs in order to extend Rayleigh's method simply to deal with extensional as well as inextensional deformation of thin shells. In the remainder of the paper we shall give some examples of this.

There remains one preliminary before we can tackle a shell problem. Consider a smooth curved surface which has principal radii of curvature R_1, R_2. The Gaussian curvature K is given by $K = 1/R_1R_2$, which is entirely consistent with the previous definition. Suppose now that the surface is deformed so that there are small changes of curvature κ_1 and κ_2 in directions 1 and 2 respectively. Then the corresponding first-order change in Gaussian curvature δK is given by

$$\delta K = \frac{\kappa_1}{R_2} + \frac{\kappa_2}{R_2}. \tag{3}$$

Note that the *change* in curvature may also involve a twisting κ_{12} with respect to the local principal axes of the undeformed surface; this however does not enter the expression for the change in Gaussian curvature. In the case of a shell which is originally in the form of a circular cylinder of radius a, (3) reduces to

$$K = \frac{\kappa_x}{a} \tag{4}$$

where κ_x is the change in curvature of a generator. Here we have written K rather than δK, as the original cylindrical surface has zero Gaussian curvature.

RADIAL POINT LOAD APPLIED TO A SPHERICAL SHELL

We shall now describe in outline the extended "Rayleigh method" calculations appropriate to a thin spherical shell of radius a and thickness t subjected to a "point load" P acting in a radial direction. The shell is made of uniform isotropic elastic material having Young's modulus E and Poisson's ratio ν. Let w be the (inward) radial displacement. Since we expect the region of interest to lie in a shallow region of the shell, we shall use a local x, y co-ordinate system, with origin at the point of application of P.

Our first requirement is a "trial displacement function" $w(x, y)$. (In the present case we would naturally choose one symmetrical about the z-axis, but we would not necessarily do this in application of the method to shells of other shapes.) Our trial function should clearly be of the form shown in Fig. 3. Since we do not know *a priori* what will be a suitable extent of the affected zone we define $\xi = x/L$, $\eta = y/L$ and put $w = w(\xi, \eta)$. L is a "size" parameter for the deflected region, having the dimension of length.

As far as strain energy of bending is concerned we treat the surface like a plate. Changes of curvature κ_x, etc. are given by

$$\kappa_x = \frac{\partial^2 w}{\partial x^2} = \frac{1}{L^2}\frac{\partial^2 w}{\partial \xi^2} \text{ etc.}$$

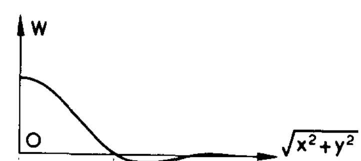

Fig. 3. Trial normal displacement function for a shallow spherical shell sustaining a normal load at the origin.

The total strain energy of bending is thus given by

$$U_B = \frac{D}{2}\iint (\kappa_x^2 + \cdots)\,dx\,dy$$

where $D = Et^3/12(1-\nu^2)$ and $(\kappa_x^2 + \cdots)$ stands for a complete expression to be given later. This may therefore be written

$$U_B = \frac{t^2}{L^2}\frac{Et}{1-\nu^2}\cdot\frac{1}{24}\iint \left(\left(\frac{\partial^2 w}{\partial \xi^2}\right)^2 + \cdots\right) d\xi\,d\eta. \tag{5}$$

The integration is performed over the entire local area.

To determine the strain energy of stretching we need to evaluate the expression

$$\frac{Et}{1-\nu^2}\frac{1}{2}\iint (\epsilon_x^2 + \cdots)\,dx\,dy.$$

To find expressions for ϵ_x etc. we use eqns (2) and (3): here $R_1 = R_2 = a$, so

$$-\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} - \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{1}{a}\frac{\partial^2 w}{\partial x^2} + \frac{1}{a}\frac{\partial^2 w}{\partial y^2}. \tag{6}$$

Thus the simplest plan in this example is to have

$$\epsilon_x = \epsilon_y = -w/a, \quad \gamma_{xy} = 0. \tag{7}$$

(Note that in using (6) we do not need to consider explicitly the tangential components of displacement. The present mode clearly corresponds to pure radial displacement).

This gives, after re-arrangement,

$$U_S = \frac{L^2}{a^2}\frac{Et}{1-\nu^2}\frac{1}{2}\iint (w^2 + \cdots)\,d\xi\,d\eta. \tag{8}$$

For a given function $w(\xi, \eta)$ our next task is to minimise the total strain energy $U = U_B + U_S$ with respect to L; having done this we shall then equate the minimised strain energy to $\frac{1}{2}Pw_0$, which will give an over-estimate (by Rayleigh's principle) of the stiffness of the structure.

Writing $U_B = (t^2/L^2)\bar{U}_B$ and $U_S = (L^2/a^2)\bar{U}_S$ we find on minimising that

$$L = (at)^{1/2}(\bar{U}_B/\bar{U}_S)^{1/4}$$

which can easily be evaluated, for a given function $w(\xi, \eta)$.

The method thus clearly reveals that the linear scale of the extent of the deformed region is of order $(at)^{1/2}$. We also note that the strain energy of stretching involves the integration of w^2 over the surface. This leads in turn to an analogy between the strain energy of stretching of the surface and strain energy of distortion of an equivalent simple elastic foundation. Therefore, the approximate solution of the problem by Rayleigh's method is directly analogous to the (exact) solution of a plate on an elastic foundation, and we may use available solutions of that problem (e.g. Timoshenko and Woinowsky Krieger[3], Chap. 8) directly, without the need to select "trial functions". We should expect the resulting stiffness to be an upper bound on the actual stiffness on account of the fact that we have used in effect an artificial (but of course kinematically admissible) mode of deformation: in the present case it is purely radial. Further study of this problem (the details of which will be given elsewhere) shows that the "plate-on-elastic-foundation" analogy is preserved even when tangential displacements are allowed; the foundation stiffness is simply Et/a^2 and the "plate-on-elastic-foundation" solution[3] is in excellent agreement with the results of numerical studies by Flügge and Elling[11].

THE "PINCHED" CYLINDRICAL SHELL

The shell and its loading are shown in Fig. 4. The shell has length $2l$, radius a and uniform thickness t. It is made of uniform isotropic elastic material having Young's modulus E and Poisson's ratio ν. (We shall later make some brief remarks about the effect of various kinds of anisotropy on the solution.) The shell is loaded by two diametrically opposed forces P applied at the mid-plane. The co-ordinate system to be used is shown in Fig. 4(a).

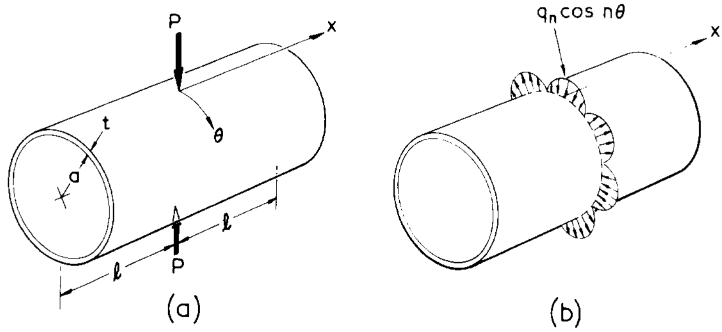

Fig. 4. (a) Cylindrical shell showing loading, dimensions and co-ordinate system. (b) A Fourier component of the loading. (The component shown is $n = 4$).

Following Rayleigh and others (e.g. Hoff[12], Holland[13]) we shall use Fourier series to decompose the problem into a set of sub-problems. The loading may thus be expressed as a radially-directed distributed load q applied around the circumference at the mid-plane:

$$q = \frac{P}{\pi a} + \frac{2P}{\pi a}\sum_{2}^{\infty} \cos n\theta \quad (n \text{ even}). \tag{9}$$

Our main task is to find the response of the shell to a single component of the loading shown in Fig. 4(b).

An obvious mode of deformation to consider involves radial displacement w of the form

$$w = w_n(x) \cos n\theta.$$

We shall use Rayleigh's method with the help of (2) and (4) to find a suitable function $w_n(x)$.

Our first task is to find the strain energy of distortion of a typical small element of the shell surface. In general, the components of surface strain at any point are ϵ_x, ϵ_θ, $\gamma_{x\theta}$ and the corresponding strain energy (of "stretching") per unit area is given by

$$\frac{Et}{2(1-\nu^2)}\left(\epsilon_x^{\,2} + \epsilon_\theta^{\,2} + 2\nu\epsilon_x\epsilon_\theta + \frac{(1-\nu)}{2}\gamma_{x\theta}^2\right). \tag{10}$$

Similarly, for components of curvature-change κ_x, κ_θ, $\kappa_{x\theta}$ the corresponding strain energy (of "bending") per unit area is given by

$$\frac{Et^3}{24(1-\nu^2)}(\kappa_x^{\,2} + \kappa_\theta^{\,2} + 2\nu\kappa_x\kappa_\theta + 2(1-\nu)\kappa_{x\theta}^2). \tag{11}$$

Having specified the form of w (the most obvious feature of a mode of deformation) we must now decide what policy to adopt in relation to the two displacement components u and v. The advantage of eqn (2) is that we need not do this directly: we can instead specify two of the strain components. The simplest possibility, which turns out as we shall see to be very satisfactory in a particular range of geometries, is to take $\epsilon_\theta = \gamma_{x\theta} = 0$. We thus obtain, from (2) and (4)

$$-\frac{1}{a^2}\frac{\partial^2 \epsilon_x}{\partial \theta^2} = \frac{w_n''(x)}{a}\cos n\theta. \tag{12}$$

As there is zero gross axial tension on the shell and we expect ϵ_x to be continuous, we can integrate (12) to give

$$\epsilon_x = \frac{a}{n^2} w_n''(x) \cos n\theta. \tag{13}$$

In order to obtain expressions for the components of curvature change we must, strictly, investigate the tangential displacement functions u, v, since these variables occur in the strain-displacement equations (Sanders[9]):

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_\theta = \frac{1}{a}\frac{\partial v}{\partial \theta} + \frac{w}{a}, \quad \gamma_{x\theta} = \frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial \theta} \tag{14}$$

$$\left.\begin{aligned} \kappa_x &= \frac{-\partial^2 w}{\partial x^2}, \quad \kappa_\theta = -\frac{1}{a^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{a^2}\frac{\partial v}{\partial \theta}, \\ \kappa_{x\theta} &= -\frac{1}{a}\frac{\partial^2 w}{\partial x \partial \theta} + \frac{3}{4a}\frac{\partial v}{\partial x} - \frac{1}{4a^2}\frac{\partial u}{\partial \theta}. \end{aligned}\right\} \tag{15}$$

Solving (14) in conjunction with (13) and $\epsilon_\theta = \gamma_{x\theta} = 0$, and using obvious conditions to provide the necessary constants of integration, we find:

$$\left.\begin{aligned} v &= \frac{-w_n(x)}{n} \sin n\theta \\ u &= \frac{-a}{n^2} w_n'(x) \cos n\theta. \end{aligned}\right\} \tag{16}$$

Substituting these expressions into (15) we obtain:

$$\left.\begin{aligned} \kappa_x &= -w_n''(x) \cos n\theta \\ \kappa_\theta &= \frac{(n^2-1)}{a^2} w_n(x) \cos n\theta \\ \kappa_{x\theta} &= \frac{(n^2-\frac{1}{2})}{n}\frac{w_n'(x)}{a} \sin n\theta. \end{aligned}\right\} \tag{17}$$

Thus, while the evaluation of "stretching" strain energy for the assumed mode is easy, the evaluation of "bending" strain energy is messy. But the problem can be simplified satisfactorily in the following way.

Consider a trial function $w_n = w_0 \cos \pi x/b$. The expressions for κ_x, κ_θ and $\kappa_{x\theta}$ take the form of a constant multiplied by $\cos n\theta \times \cos \pi x/b$ or $\sin n\theta \times \sin \pi x/b$. The *amplitudes* of the three expressions are as follows:

$$\left.\begin{aligned} &\kappa_x: \frac{\pi^2}{b^2} w_0 \\ &\kappa_\theta: \left(\frac{n^2-1}{a^2}\right) w_0 \\ &\kappa_{x\theta}: \left(\frac{n^2-\frac{1}{2}}{n}\right)\frac{\pi}{ba} w_0. \end{aligned}\right\} \tag{18}$$

Now b is a longitudinal half wavelength, and the corresponding circumferential half-wavelength, say c, is equal to $\pi a/n$; so the three amplitudes may be expressed as

$$\left.\begin{aligned} &\frac{\kappa_x}{w_0}: \frac{\pi^2}{b^2} \\ &\frac{\kappa_\theta}{w_0}: \frac{\pi^2}{c^2} - \frac{1}{a^2} \\ &\frac{\kappa_{x\theta}}{w_0}: \frac{\pi^2}{bc} - \frac{1}{2}\frac{c}{a^2 b}. \end{aligned}\right\} \tag{19}$$

Consequently if $b \gg c$ the bending strain energy is dominated by the κ_θ term, and conversely if $b \ll c$ the κ_x term dominates. Only in the region $b \simeq c$ does the term $\kappa_{x\theta}$ dominate. The expressions, and the conclusions, are virtually the same as for the bending of a simply-supported rectangular plate of dimensions b, c in a single half wave in each direction: the bending strain energy is dominated by bending across the smaller dimension except for nearly square plates, when twisting plays a major part.

Thus there is a possibility that in the evaluation of the bending strain energy either the κ_θ or the κ_x term will be the only significant one. Of course, it is unlikely that the best function $w_n(x)$ will be simply sinusoidal, so this conclusion may be unwarranted in general. However, a detailed study by A. C. Smith on the "homogeneous" case of a thin cylindrical shell with zero surface tractions shows that the two families of roots of the characteristic equations (see Hoff [2] or Holland [13]) correspond to the strain energy being dominated by *either* longitudinal stretching and circumferential bending ("long-wave" solutions) *or* circumferential stretching and longitudinal bending ("short-wave" and axi-symmetric solutions). The two families merge in the region where both the circumferential and longitudinal strain energy of stretching cease to be important, i.e. the behaviour approximates that of a flat plate.

There is no need to investigate this point in any more detail here. We shall simply take the suggestion from Smith's work that *either* circumferential *or* longitudinal bending dominates the strain energy of bending, and use it as a *hypothesis* to be tested later. In the present example we have already chosen a mode of stretching in which only the longitudinal strains are non-zero, so we shall try first the hypothesis that in bending the circumferential change of curvature dominates. This will lead to a solution, and we shall be able to investigate later the magnitude of bending in the longitudinal direction. We shall see that except in certain cases the strain energy due to longitudinal bending is negligible: the exceptional cases provide certain limits to the applicability of the solution. Next we shall try the hypothesis that longitudinal bending dominates, and find that it immediately leads to a contradiction which is sufficient to exclude it from further consideration.

Taking κ_θ as the dominant bending term, and ϵ_x as already calculated (eqn (13)) we have the following approximate expression for strain energy per unit area of shell:

$$\frac{Et}{2(1-\nu^2)}\frac{a^2}{n^4}\cos^2 n\theta(w_n''(x))^2 + \frac{Et^3}{24(1-\nu^2)}\frac{(n^2-1)^2}{a^4}\cos^2 n\theta(w_n(x))^2. \tag{20}$$

Our aim, of course, is to find the function $w_n(x)$ which minimises, or nearly minimises, the total potential energy of the system, i.e. the sum of the strain energy and the potential energy of the load. As the load is $q_n \cos n\theta$ applied around the circumference at the plane $x = 0$, the potential energy per unit circumference is $-q_n w_n(0)\cos^2 n\theta$. It is easy to integrate all of these functions around the circumference, as the mean value of $\cos^2 n\theta$ is 0.5. Consequently the total potential energy to be minimised is:

$$\frac{Et}{1-\nu^2}\frac{a^2}{n^4}\frac{\pi a}{2}\int_{-l}^{l}(w_n''(x))^2\,dx + \frac{Et^3}{12(1-\nu^2)}\frac{(n^2-1)^2}{a^4}\frac{\pi a}{2}\int_{-l}^{l}(w_n(x))^2\,dx - \pi a q_n w_n(0). \tag{21}$$

Now in this expression the displacement function $w_n(x)$ only appears as such and in its second derivative. This immediately suggests an analogy with a *beam-on-elastic foundation* problem, shown in Fig. 5(a). There is an elastic beam of flexural rigidity B and length $2l$, resting on an elastic foundation of stiffness m, and loaded centrally by a force Q. For a trial displacement function $w(x)$ the strain energy per unit length of foundation is $(m/2)(w(x))^2$, and the strain energy of bending per unit length of beam is $(B/2)(w''(x))^2$. Consequently we seek a function $w(x)$ which minimises the expression

$$\frac{B}{2}\int_{-l}^{l}(w''(x))^2\,dx + \frac{m}{2}\int_{-l}^{l}(w(x))^2\,dx - Qw(0). \tag{22}$$

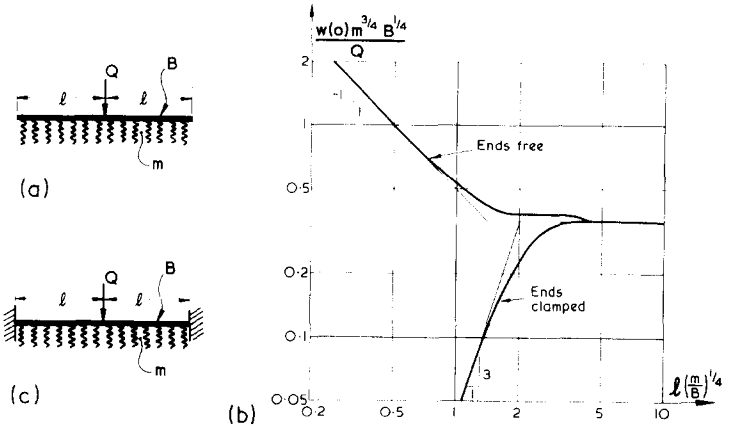

Fig. 5. Beam-on-elastic foundation results for centrally loaded beam (after Hetényi). (b) is a logarithmic plot. The light lines are asymptotes which correspond to simple special. $w(0)$ is the central displacement.

Clearly there is a direct analogy between the two cases, which may be expressed as follows:

$$\left.\begin{aligned} B &\leftrightarrow \frac{\pi E t a^3}{(1-\nu^2)n^4} \\ m &\leftrightarrow \frac{\pi E t^3(n^2-1)^2}{12(1-\nu^2)a^3} \\ Q &\leftrightarrow \pi a q_n. \end{aligned}\right\} \tag{23}$$

Note that the bending stiffness of the analog beam derives from longitudinal stretching of the shell, while the stiffness of the analog foundation derives from circumferential bending of the shell. This is not altogether surprising: in the case $n = 1$ (which we shall not need to use in the problem under consideration), B is simply the "EI" value for a tube in pure flexure (except for the term $(1-\nu^2)$, which we shall discuss later), and $m = 0$, as the cross-sectional shape does not change in this mode.

Of course, now that we have reduced the problem to the beam-on-elastic foundation, we can simply look up whatever results we need in Hetényi[6]. In the present case the central deflection $w(0)$ is given by

$$w(0) = \frac{Q}{2\sqrt{2}m^{3/4}B^{1/4}} \cdot \frac{\cosh\psi + \cos\psi + 2}{\sinh\psi + \sin\psi}$$

where

$$\psi = l\sqrt{2}(m/B)^{1/4}$$

which is plotted in dimensionless form in Fig. 5(b). For sufficiently small values of ψ the stiffness $Q/w(0)$ is equal to $2lm$; the beam remains virtually straight. This corresponds exactly to an inextensional mode for the shell. On the other hand, for large values of ψ the stiffness is independent of ψ; this corresponds to a long beam, with deflection localised to the central region. Note that there is a relatively short transition region between these two extreme cases. It is tempting to think of "long" and "short" beams, but this can be misleading: the proper parameter is $l(m/B)^{1/4}$. In the present context there is a further complication, because both B and m depend on the value of the mode number n. Figure 6 presents the data of Fig. 5(b) transformed into "shell" variables; the "mode flexibility" $w_n(0)/q_n$ is a function of mode number n. For sufficiently small mode numbers the behaviour corresponds to "short beam" i.e. "inextensional" deformation, while for large mode numbers the deformation is localised. The value, n^*, of n at which the two asymptotes cross (which is found from the date on Fig. 5(b) using values of B and m from (23)) turns out to be proportional to $\lambda^{-1/2}$, where λ is a dimensionless length defined by

$$\lambda = lt^{1/2}/a^{3/2}. \tag{24}$$

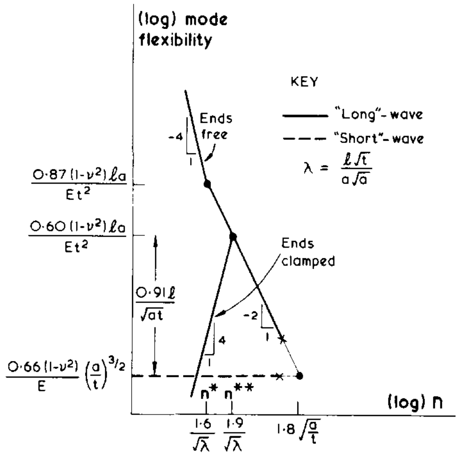

Fig. 6. Schematic plot of mode flexibility ($= w_n(0)/q_n$) against mode number n. The diagram is not to scale, but salient points (where asymptotes intersect) are indicated. The lines of slope -4, -2 and $+4$ correspond to lines on Fig. 5 of slope $-1, 0, +3$, respectively. The "short-wave" mode flexibilities are independent of n and are in most cases stiffer (i.e. less flexible) than their long-wave counterparts. Smooth transition-curves between asymptotes (see Fig. 5) have been omitted. The slopes are all integral because terms (n^2-1) have been replaced by n^2. X marks the points at which the simple formulas begin to be inaccurate.

For simplicity in plotting Fig. 6 we have replaced the term (n^2-1) wherever it appears by n^2. This discrepancy can be allowed for easily in doing the calculations for a given shell.

We must next investigate the magnitude of the longitudinal bending strain energy term which we discarded earlier in the interests of simplicity. To the expression (20) should be added the term

$$\frac{Et^3}{24(1-\nu^2)}\cos^2 n\theta(w_n''(x))^2. \tag{25}$$

This can conveniently be added in to the "longitudinal stretching" term, where it has the effect of modifying the value of B, and we could if we wished re-define B and alter Fig. 6 accordingly. But there are, of course, other terms in the expression for bending strain energy (eqn (11)) which we have also neglected, and it is more satisfactory to proceed by noting that the value of B will be altered by less than 10% on account of inclusion of longitudinal bending provided $n < (a/t)^{1/2}$, approximately.

This furnishes the limit marked X in Fig. 6. The exact layout of Fig. 6 thus depends both on the value of λ and of a/t in a given problem; the general character is of course the same.

Having started with a "longitudinal stretching" mode we first examined the hypothesis that the change of circumferential curvature provided the dominant term in the bending strain energy expression. If instead we make the hypothesis that *longitudinal* bending is dominant, the effect is to replace the second term in (20) by term (25). We see immediately that this simply eliminates the "elastic foundation" term in our analogy, which is absurd, as equilibrium is not possible. Consequently we reject this hypothesis.

"SHORT WAVE-LENGTH" MODES

Let us return to the point (eqn (12)) where we decided to make $\epsilon_\theta = \gamma_{x\theta} = 0$, and put instead $\epsilon_x = \gamma_{x\theta} = 0$ as a way of determining a new trial mode. Corresponding to (12) we now have

$$\frac{-\partial^2\epsilon_\theta}{\partial x^2} = \frac{w_n''(x)}{a}\cos n\theta \tag{26}$$

which gives, on integration and substitution of appropriate boundary conditions,

$$\epsilon_\theta = \frac{w_n(x)}{a}\cos n\theta. \tag{27}$$

It follows from the strain-displacement relations (14) that $u = v = 0$, so we obtain in place of (17) the following:

$$\left.\begin{aligned}\kappa_x &= -w_n''(x)\cos n\theta\\ \kappa_\theta &= \frac{n^2}{a^2}w_n(x)\cos n\theta\\ \kappa_{x\theta} &= n\frac{w_n'(x)}{a}\sin n\theta.\end{aligned}\right\} \tag{28}$$

These expressions differ by little, particularly for large values of n, from the previous ones. Again we make two separate hypotheses that κ_θ and κ_x dominate the strain energy of bending. If κ_θ dominates we find an energy expression in which B, the bending stiffness in the beam analog, is zero; as this leads to absurdities we do not proceed further. On the other hand, if κ_x dominates, we again find an energy expression corresponding to a beam on elastic foundation, with

$$\left.\begin{aligned}B &\leftrightarrow \frac{\pi E t^3 a}{12(1-\nu^2)}\\ m &\leftrightarrow \frac{\pi E t}{(1-\nu^2)a}.\end{aligned}\right\} \tag{29}$$

Note that n does not appear in these expressions, and the characteristic length scale is precisely as for axially symmetric deformation, viz $(at)^{1/2}$. The "mode flexibility", shown in Fig. 6, is independent of n, but when the value of n is sufficiently large the circumferential bending effect becomes significant. The point at which the circumferential bending effect alters the foundation stiffness by 10% is again $n \simeq (a/t)^{1/2}$, and is shown by X on the diagram.

Of the two kinds of solution (with longitudinal stretching or circumferential stretching, respectively, predominating) which we have investigated it is clear that the latter gives a lower modal flexibility in virtually all cases and so must be rejected, by Rayleigh's principle. In particular, the constant term on the R.H.S. of (9) produces negligible deflection.

The region where the mode stiffnesses are of comparable magnitude is precisely the region where our two hypotheses break down. Further investigation shows that for high values of n stretching of the surface produces insignificant strain energy in comparison with bending, and the behaviour in fact approximates that of a flat plate. This régime cannot satisfactorily be investigated in terms of "beam on an elastic foundation".

It is now a simple matter to superpose the effects of various terms in the Fourier series for a shell of given geometry. It is clear that the fundamental geometrical parameter is λ, and the results are shown in Fig. 7. For a "short" shell—i.e. one with a small value of λ—the value of n^* is relatively large (see Fig. 6) so for practical purposes all of the modes are inextensional. Summation of the appropriate series (to infinity) gives precisely the result quoted by Timoshenko [3] for the inextensional case; the radial displacement δ on the point of application of load is given by

$$\delta = \frac{1}{2.24}\frac{P(1-\nu^2)}{Ea}\left(\frac{a}{t}\right)^{5/2}\frac{1}{\lambda}. \tag{30}$$

At the other extreme—large values of λ—all of the modes correspond to "long beam" cases and the summation is again straightforward. Deflection under the load (which is of course independent of λ) is given by

$$\delta = \frac{1}{1.23}\frac{P(1-\nu^2)}{Ea}\left(\frac{a}{t}\right)^{5/2}. \tag{31}$$

For intermediate values of λ the modes with lower values of n are inextensional while those for higher values of n are "long beam" cases. The summation produces a relatively short transition between the two extremes, as shown in Fig. 7.

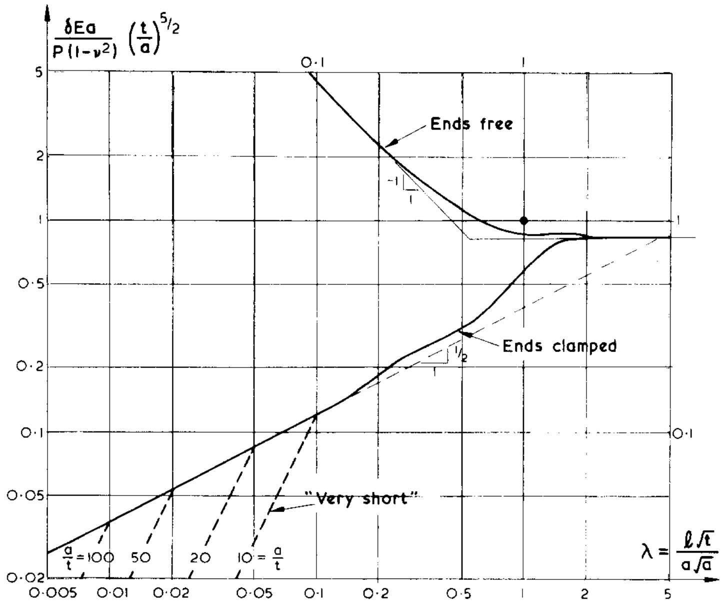

Fig. 7. Logarithmic plot of deflection coefficient against length coefficient for the pinched cylindrical shell (Fig. 4a). δ is the radial displacement of the point of application of load P. Deflections have been obtained by summation of Fourier terms using the results of Fig. 6, with transition curves as in Fig. 5. The broken lines correspond to "flat plate" results, and the form of the transition has not been investigated.

THE EFFECT OF CLAMPED EDGES

So far we have considered a pinched cylindrical shell with free ends. It is not difficult to extend the method to deal with other boundary conditions, for example clamped edges. It is easy to show that for each mode this corresponds simply to clamping the ends of the analogous beam on elastic foundation, Fig. 5(c). The principal consequence is that for short beams the stiffness is virtually the same as for a clamped-ended beam having no elastic foundation, as shown in Fig. 5(b) and this in turn leads to a modal flexibility for the shell proportional to λ^3, as shown in Fig. 6. This situation corresponds almost to "membrane" theory, in which bending plays no part in carrying the load. Our method is of course an approximate one, and does not consider the fine detail of the zone immediately under the line of application of the load; this zone is an awkward one in membrane theory.

Whereas for the cylindrical shell with free ends the modal flexibility decreases with n irrespective of the regime of behaviour (Fig. 6), for clamped ends the situation is different: there is a "most flexible mode" at about $n = n^{**} = 1.9\lambda^{-1/2}$. But n is by definition an even integer, so we find for large values of λ that mode $n = 2$ is most flexible and dominates the deflection. In the region of $\lambda = 0.25$, $n = 4$ is the most flexible mode, and its contribution produces the "bump" in the curve in Fig. 7. The curve has been obtained by summation of appropriate terms, taking into account the detailed lower "transition curve" of Fig. 5(b). For smaller values of λ the deflection is not so strongly dominated by the term for the most flexible mode, and it may be shown by summation of a simple series that for small values of λ the asymptote is given by

$$\delta = \frac{1}{2.56}\frac{P(1-\nu^2)}{Ea}\left(\frac{a}{t}\right)^{5/2}\lambda^{1/2}. \tag{32}$$

There is an additional complication with clamped shells. For *very* short shells the value of n^{**} may be higher than $(a/t)^{1/2}$, so the previous calculation would be invalid. Shells as short as this behave like a long flat plate of width $2l$. As previously noted, our "beam on elastic foundation" analogy is inadequate in this region. From Timoshenko[3] we find that the deflection under the point of application of load is given by

$$\delta = \frac{Pl^2(1-\nu^2)}{2.87Et^3}. \tag{33}$$

This relationship is also plotted on Fig. 7. The parameter λ is no longer particularly useful, and lines for different values of a/t are given. This part of the diagram is re-plotted more effectively by the use of the length parameter $l/(at)^{1/2}$ in Fig. 8. Also on this diagram is given the corresponding lines for a clamped cap of a spherical shell of radius a and thickness t, where l is now the outer radius of the cap. For sufficiently small values of $l/(at)^{1/2}$ we use the result for a clamped circular plate, while for the other asymptote we give the result for the case where the zone of deformation is localised, obtained by the method outlined in an earlier section. This diagram points out an important difference between the two kinds of shell: we can think of purely localised deformation of a spherical shell, but not for a cylindrical shell with clamped edges where $l/(at)^{1/2} > 1.5$, say.

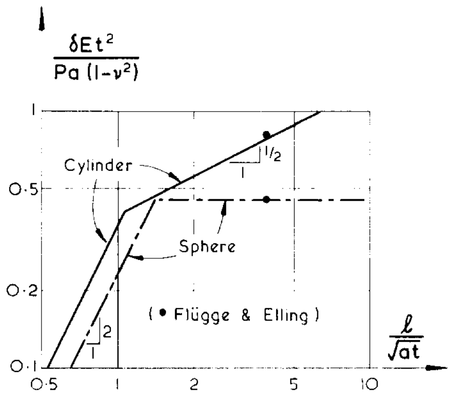

Fig. 8. Replot of the lower left-hand corner of Fig. 7, using the parameter $l/\sqrt{at}$. Corresponding results for a spherical shell (clamped at distance l from the loaded point) are also shown. The two points represent numerical solutions from Ref. [11], in which $\nu = 0.3$.

To fill in the "transition" curves in Fig. 8 (and the corresponding parts of Fig. 7) would require more sophisticated techniques than those described here. Even the transition curve for a spherical shell (Fig. 8), which may be done analytically, involves laborious calculation. But the incompleteness of the curves in Figs. 7 and 8 should not detract from what is perhaps the main outcome of this work, namely the description of distinct regimes of structural behaviour, depending on the values of two different geometrical groups.

This is summarised in Table 1. In any given case the first step is to work out the value of $l\sqrt{t}/a\sqrt{a}$ and also, if this is less than 1 and the shell is clamped, the value of $l/\sqrt{at}$. In a case where either of these is a "borderline value", Figs. 7 and 8 should be consulted. The two groups are not new in themselves; see, e.g. Chap. 7; on buckling of cylindrical shells, in Flügge [4]. Indeed, it might be argued that Poisson's ratio should enter the groups, and we can only plead

Table 1.

End conditions	Length	Mode	Length criterion	Appropriate equation
free	"short"	inextensional	$\frac{l\sqrt{t}}{a\sqrt{a}} < 0.5$	30
	"long"	bending/stretching	$\frac{l\sqrt{t}}{a\sqrt{a}} > 1$	31
clamped	"long"	bending/stretching	$\frac{l\sqrt{t}}{a\sqrt{a}} > 2$	31
	"short"	mainly membrane	$\frac{l\sqrt{t}}{a\sqrt{a}} < 1$ and $\frac{l}{\sqrt{at}} > 1$	32
	"very short"	flat-plate	$\frac{l}{\sqrt{at}} < 1$	33

that our energy method gives slight inaccuracies in this respect; see later. In any case this is surely nòt a matter of great concern to engineers.

OTHER FEATURES OF THE METHOD

There are many interesting features of the method which we have not described. For present purposes it should be sufficient to give short comments on some of them.

In the examples the ends of the cylindrical shell were either completely free or clamped. Other boundary conditions for the shell are analagous to distinct boundary conditions for the beam. For example, if the end of the shell is held circular but not restrained axially, the corresponding beam has a simply supported end. As we have seen, bending moments in the analogous beam correspond to stressing of the shell in the axial direction. Boundary conditions involving axial load (like those studied experimentally by Kildegaard[14]) transform to moments applied at the end of the analogous beam.

Our method correctly reproduces the inextensional modes for shells with small values of λ, even for low values of n. As is well known, the simplified equations of Donnell[15] give slightly erroneous results in this region; using the kind of approximation employed by Donnell, we would have to replace the term (n^2-1) in our expression for m (eqn (23)) by the not-so-accurate n^2. Of course, for high values of n the difference is negligible.

In our work we have not met directly an objection which Love raised to Rayleigh's original work. At a completely free edge $M_x = 0$, and this requires, via the elastic law $\kappa_x = -\nu\kappa_\theta$. Inextensional modes for cylindrical shells of course have $\kappa_x = 0$. Bassett[16] and Lamb[17] pointed out the nature of the "boundary layer" for such shells. In our work shortwave solutions are closely related to boundary-layer effects. Further investigation shows that the high *stiffness* of these modes is responsible for their having little impact on the gross distortion of the shell.

A detailed investigation shows that many of the expressions developed by means of our energy approach are very nearly exact. Our beam-on-elastic foundation analog would become nearly perfect if the term $(1-\nu^2)$ in the expressions for B (eqn (23)) and m (eqn (29)) were replaced by 1. This factor is attributable to the rather crude way in which we had only one of the three strain components ϵ_θ, ϵ_x, γ_{xy}, non-zero at a time. It is possible to develop more subtle constraints on surface stretching which do not have an appreciable effect on the strain energy of bending but which remove the factor $(1-\nu^2)$.

Ashwell and Sabir[18] and Morley[19] have used a "pinched" cylindrical shell with free ends as a test-problem for finite element calculations on shells. All calculations so far reported by these authors have been for short shells with $\lambda \leq 0.29$, i.e. in the almost purely inextensional régime: the results are in excellent agreement with the data of Fig. 7.

A strong feature of the method is that it can easily be adapted to deal with various kinds of anisotropy, particularly those arising from the application of circumferential or longitudinal stiffening ribs sufficiently closely spaced for "smearing" to be an appropriate form of analysis. For example, circumferential ribs may give a big boost to the circumferential bending stiffness while having very little effect on longitudinal stretching. In the present case they would have more effect in stiffening up a short cylinder (small λ) than longitudinal ribs. The details can be worked out easily.

Lastly we remark that A. C. Smith has done careful experiments on the deformation of model shells subjected to point loads, and has found direct experimental confirmation for "mode flexibility factors" of the kind shown in Fig. 6. This work will be reported elsewhere.

CONCLUSION

Two sets of conclusions may be drawn from this work. First, we can see that the kinematic relations (1) and (2) are immediately useful. Not only do they enable us to extend Rayleigh's method easily and naturally to non-inextensional problems but they also give clear insight into the kinematics of deformation of shells which is advantageous in other aspects of analysis including plasticity, vibrations, and buckling. Examples will be reported elsewhere.

Second, we note that the method leads directly to appropriate dimensionless groups which help to delineate distinct regimes of the structural behaviour of shells. This practical outcome will commend the method to engineers engaged in the design of shells and also to those involved in planning experiments and correlating existing experimental evidence.

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